On the scale-invariant distribution of the diffusion coefficient for classical particles diffusing in disordered media

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Abstract. The scaling form of the whole distribution $P(D)$ of the random diffusion coefficient $D(x)$ in a model of classically diffusing particles is investigated. The renormalization group approach above the lower critical dimension $d = 0$ is applied to the distribution $P(D)$ using the n-replica approach. In the annealed approximation ($n = 1$), the inverse Gaussian distribution is found to be the stable one under rescaling. This identification is based on symmetry arguments and subtle relations between this model and that of fluctuating interfaces studied by Wallace and Zia. The renormalization-group flow for the ratios between subsequent cumulants shows a regime of pure diffusion for small disorder, where $P(D) \to \delta(D - \bar{D})$, and a regime of strong disorder in which the cumulants grow infinitely large and the diffusion process is ill defined. The boundary between these two regimes is associated with an unstable fixed-point and subdiffusive behaviour $\langle x^2 \rangle \sim t^{1-d/2}$. For the quenched ($n \to 0$) case we find that unphysical operators are generated raising doubts on the renormalizability of this model. Implications for other random systems near their lower critical dimension are discussed.

1. Introduction

The asymptotic behaviour of classical particles diffusing in a disordered media has been the focus of many recent investigations (for a recent review see [1]). The models which have attracted most attention are formulated in terms of random transition probabilities or local random quenched forces. In most of these problems $d = 2$ is a special critical dimension and most attention has been focused on the anomalous diffusion which usually occurs for $d < 2$ [2–7]. The anomalous diffusion is manifested by the long-time behaviour of the form:

$$\langle x^2(t) \rangle \sim t^{2\mu} \tag{1}$$

with $2\mu < 1$ ($2\mu > 1$), the behaviour is said to be sub- (super-) diffusive. Attention has also been devoted to a simpler model in which the diffusion constant $D(x)$ is a local quenched random variable and the diffusion equation for the local density $\rho(x,t)$ is

$$\frac{\partial \rho(x,t)}{\partial t} = \nabla \cdot [D(x) \nabla \rho(x,t)]. \tag{2}$$

This equation also describes the continuum limit of a random resistor network in which $D(x)$ is the local conductance. In this context one can define a Hamiltionian

$\mathcal{E} = \frac{1}{2} \int d^d x \, D(x) (\nabla V(x))^2$, where $V(x)$ is the local voltage. In the physical problem in which the $D(x)$ are themselves fluctuating dynamical variables the partition function (annealed average) required averaging, while if the $D(x)$ are frozen, it is the free energy which should be averaged (quenched average).

The local $D(x)$ are expressed as a sum of a uniform part $\check{D}$ and a deviations part $\Delta D(x)$:

$$D(x) = \check{D} + \Delta D(x) = \check{D} + 2\delta D(x) \tag{3}$$

$$\langle \delta D(x) \delta D(y) \rangle = \langle \delta D^2 \rangle \delta(x - y) \tag{4}$$

(where $\delta D(x) = \Delta D(x)/2$ was introducted for computational convenience). This model was considered somewhat trivial because for weak disorder, $\Delta D / \check{D} \ll 1$, there is an asymptotic normal diffusive behaviour at any dimension $d > 0$ (the irrevlevance of the disorder near the pure diffusive fixed-point is explicitly given later).

However, this does not preclude a non-trivial behaviour for the strong disorder regime. The existence of such a regime will be manifested by a finite basin of attraction of the pure diffusive fixed-point in the space of all possible probability distribution $P(D)$ for the locally uncorrelated diffusion constants. This space may be represented by the couplings $g^{(k)}$ related to the cumulants of the distribution by

$$\langle \delta D(x) \delta D(y) \rangle_c = 2! g^{(2)} \delta(x = y)$$

$$\langle \delta D(x) \delta D(y) \delta D(z) \rangle_c = 3! (-1) g^{(3)} \delta(x = y = z)$$

$$\cdots \quad \cdots$$

$$\langle \delta D(x_1) \cdots \delta D(x_k) \rangle_c = k! (-1)^k g^{(k)} \delta(x_1 = x_2 = \cdots = x_k). \tag{5}$$

The central question we set ourselves to address in this work is whether there exists a non-trivial distribution that will be associated with the 'critical' behaviour on the separatrix bordering this basin of attraction in the cumulant's infinite-dimensional space. There may be more than one such distribution or even a whole family of them depending on the renormalization-group (RG) flow on the separatrix ('critical manifold') itself.

However, even the existence of such one scale-invariant distribution is far from trivial since the theory may be either non-renormalizable (if an infinite number of new relevant couplings are generated) or renormalizable in a larger 'unphysical' space (if relevant operators, distinct from the above cumulants, are generated).

Similiar questions have recently arisen in the study of quantum diffusion (localization) where it was shown that the theory is renormalizable if the whole distribution of the conductance is considered [8–10]. The distribution takes three different forms depending on whether the system is metallic, insulating or at criticality in between [8]. Similar questions of renormalizability always arise when a random system is considered near its lower critical dimension. For non-random models symmetry ensures full renormalizibity near the lower critical dimension in which they have an infinite number of marginal operators mixed by the RG flows (e.g. $O(M)$ symmetry [11] for the M-component Heisenberg model in $d = 2 + \epsilon$). For random systems the effective replicated Lagrangian has a larger number of couplings without having larger symmetries to impose constraints on the renormalized couplings to be related to each other (as to preserve these symmetries). So the question of whether

random systems are renormalizable near their lower critical dimension is much more delicate. While the quantum diffusion in $d = 2 + \epsilon$ is an example of a renormalizable theory [8–10], the random-field $O(M)$ model in $d = 4 + \epsilon$ provides a counterexample [12].

We therefore hope that our systematic study of another such system will help to shed more light on this puzzling question.

The paper is organized as follows. In the next section (2) the n-replicated Lagrangian will be derived. From this Lagrangian, which has the cumulants of the initial distribution related to its bare couplings, all correlation functions may be derived. In section 3 we analyse the so-called annealed approximation ($n = 1$) first by the standard RG approach and then by a search of the universal distribution based on symmetry. Section 4 is devoted to the study of the quenched average ($n \to 0$). The problems encountered are discussed. In section 5 we summarize the implications of these investigations for the diffusion problem, in particular, and for random systems in general.

2. The replicated Lagrangian

Without losing generality, we assume all the particles to be at $x = 0$ at the initial time $t = 0$, namely

$$\rho(x, 0) = \delta(x). \tag{6}$$

The Laplace transform of $\rho(x, t)$ is defined as

$$\bar{\rho}(x, m^2) = \int_0^\infty \mathrm{d}t \, \rho(x, t) \mathrm{e}^{-m^2 t}. \tag{7}$$

$\bar{\rho}(x, m^2)$ obeys the equation

$$m^2 \bar{\rho}(x, m^2) - \rho(x, 0) = \nabla \cdot [D(x)\nabla]\bar{\rho}(x, m^2) \tag{8}$$

or

$$m^2 \bar{\rho}(x, m^2) - \nabla \cdot [D(x)\nabla]\bar{\rho}(x, m^2) = \delta(x). \tag{9}$$

We may identify $\bar{\rho}(x, m^2)$ with the Green function

$$\bar{\rho}(x, m^2) = \bar{G}(x, m^2) \tag{10}$$

which is the Laplace transform of the time domain Green function

$$G(x, t) = \langle \rho(x, t)\rho(0, 0) \rangle. \tag{11}$$

All scaling properties may be extracted from the large x and t (or small m^2) behaviour of these Green functions. To be able to average these correlation functions over the realizations of the disorder we first express them as a functional integral in the standard form:

$$\bar{G}(x, m^2) = \frac{\int \mathrm{D}\phi \, \phi(x)\phi(0)\mathrm{e}^{-\mathcal{L}[\phi]}}{\int \mathrm{D}\phi \, \mathrm{e}^{-\mathcal{L}[\phi]}} \tag{12}$$

where

$$\mathcal{L}[\phi] = \int \mathrm{d}^d x \phi [\tfrac{1}{2} m^2 - \nabla \cdot (\tfrac{1}{2} D(x) \nabla)] \phi \tag{13}$$

or, after integration by parts and neglect of an unimportant boundary term,

$$\mathcal{L}[\phi] = \int \mathrm{d}^d x \{\tfrac{1}{2} m^2 \phi^2 + \tfrac{1}{2} D(x) (\nabla \phi)^2\}. \tag{14}$$

The averages over the disorder may be obtained by utilizing the replica trick. The n-replicated partition function is obtained by including n fields $\phi^\alpha(x)$, ($\alpha = 1, 2, \dots, n$):

$$Z^n = \int \prod_\alpha \mathrm{D} \phi^\alpha(x) \mathrm{e}^{-\mathcal{L}[\phi^\alpha]} \tag{15}$$

with

$$\mathcal{L}^n[\phi^\alpha] = \int \mathrm{d}^d x \left\{ \sum_\alpha [\tfrac{1}{2} m^2 (\phi^\alpha)^2 + \tfrac{1}{2} D(x) (\nabla \phi^\alpha)^2] \right\}. \tag{16}$$

Since $\lim_{n \to 0} \langle Z^n \rangle_{\mathrm{disorder}} = 1$, we may eliminate the denominator in equation (12), and calculate the average Green function from

$$\langle \bar{G}(x, m^2) \rangle_{\mathrm{disorder}} = \lim_{n \to 0} \int \prod_\alpha \mathrm{D} \phi^\alpha(x) \phi^1(x) \phi^1(0) \mathrm{e}^{-\mathcal{L}^n[\phi^\alpha]}. \tag{17}$$

Since the quenched average is usually difficult to perform the 'annealed' approximation is often used. It consists in averaging independently the numerator and the denominator in equation (12). This is also equivalent to keeping a single replica or averaging Z^n in equation (15) with $n = 1$. Therefore we shall, in the rest of this section, analyse the generalized field theories with $\mathcal{L}^n[\phi^\alpha]$ for any n. The results of specific cases of annealed ($n = 1$) and quenched ($n \to 0$) averages are discussed in the next two sections. Let us use the vector notation $\phi = (\phi^1, \phi^2, \dots, \phi^n)$. The replicated Lagrangian may be separated into a free part and an interacting part, $\mathcal{L}^n = \mathcal{L}_0 + \mathcal{L}_{\mathrm{in}}$ where

$$\mathcal{L}_0 = \frac{1}{2} \int \mathrm{d}^d x \{\tilde{D}(\nabla \phi)^2 + m^2 (\phi)^2\} \tag{18}$$

and

$$\mathcal{L}_{\mathrm{in}} = \int \mathrm{d}^d x \{\delta D(x) (\nabla \phi)^2\}. \tag{19}$$

The average over the disorder yields

$$\langle \mathrm{e}^{-\mathcal{L}_{\mathrm{in}}} \rangle_{\mathrm{disorder}} = \exp \left(\int \mathrm{d}^d x \sum_{k=2}^\infty g^{(k)} [(\nabla \phi)^2]^k \right) \tag{20}$$

where $g^{(k)}$ are the couplings defined in equation (5). We therefore have to apply the RG analysis to the following averaged partition function:

$$\langle Z^n \rangle_{\text{disorder}} = \int \prod_\alpha D\phi^\alpha(x) \exp\left\{-\int d^d x (\tfrac{1}{2}[m^2\phi^2 + \bar{D}(\nabla\phi)^2] - \sum_{k=2}^\infty g^{(k)}[(\nabla\phi)^2]^k)\right\}$$

(21)

for simplicity we shall choose $\bar{D} = 1$ in the forthcoming calculations. Also in the renormalization scheme we choose to keep this term constant. To make it dimensionless ϕ should carry dimension of $[\phi] = L^{1-d/2}$. Hence $[(\nabla\phi)^2] = L^{-d}$, and $[g^{(k)}] = L^{(k-1)d}$.

Therefore under rescaling $L \to L/b$, $g^{(k)} \to b^{-(k-1)d}g^{(k)}$ and $g^{(k)}$ are irrelevant near the free Gaussian theory for any dimension $d > 0$. We also observe that $d = 0$ is the critical dimension at which these couplings become marginal. The natural small parameter will thus be $\epsilon = d$. In the next sections we go beyond the dimensional analysis by utilizing first the RG approach and then an alternate approach based on possible symmetries of the non-trivial fixed-point distribution.

3. The annealed ($n = 1$) approximation

We begin our analysis with the consideration of the simpler annealed approximation which corresponds to the single-component field ϕ. We begin by demonstrating the problems that arise in taking the RG route and then show how these difficulties may be circumvented to find the scale-invariant distribution.

3.1. RG approach

To go beyond the naive dimensional analysis we need to expand $\exp\{-\sum_k g^{(k)} [(\nabla\phi)^2]^k\}$ in a power series, separate and integrate momenta $\Lambda/b < q < \Lambda$ (where $\Lambda = 1/a$ is the boundary of the Brillouin zone) using e^{-H_0}, re-exponentiate, and rescale all momenta $q \to bq$ so as to obtain a new effective Lagrangian. This can be done diagrammatically and here we only give the recursion relations for $k = 2, 3$ to order one loop ($l = \ln b$):

$$\frac{dg^{(2)}}{dl} = -\epsilon g^{(2)} + 12g^{(3)} + 56(g^{(2)})^2$$

(22)

$$\frac{dg^{(3)}}{dl} = -2\epsilon g^{(3)} + 240g^{(3)}g^{(2)} + 24g^{(4)} + 416(g^{(2)})^3.$$

(23)

Similar expressions may be derived for all $g^{(k)}$. They will take the form

$$\frac{dg^{(k)}}{dl} = \beta_\epsilon^{(k)}(g^{(2)}, g^{(3)}, \ldots, g^{(k+1)}) = -(k-1)\epsilon g^{(k)} + \beta_0^{(k)}(g^{(2)}, g^{(3)}, \ldots, g^{(k+1)})$$

(24)

where $\beta_0^{(k)}$ is the same beta function calculated at $\epsilon = 0$. Since $\beta_\epsilon^{(k)}$ have both positive and negative terms each of them has zeros. The challenge is to find at least

one common zero, $g^{(k)} = g^{(k)*}$ for all k, where they all vanish simultaneously. We also note that the gs will be generated under the RG recursions once the bare value of one of them is not zero. However, different couplings (not cumulants of $P(D)$) will not be generated and the space of all gs is closed under the RG transformation. These equations also tell us that at the fixed-point $g^{(k)*} \sim \epsilon^{k-1}$ and we may express them in terms of coefficients a_k such that

$$g^{(k)*} = a_k \epsilon^{k-1}. \tag{25}$$

Using the equations $\beta_\epsilon^{(k)} = 0$ all a_k may, iteratively, be related to a_2, e.g.

$$a_3 = \tfrac{1}{12} a_2 (1 - 56 a_2)$$
$$a_4 = \tfrac{1}{12} a_3 - 10 a_3 a_2 - \tfrac{52}{3} a_2^3 \qquad \text{etc.}$$

We can also deduce some information on the behaviour away from the fixed-point. Suppose we try to rescale all $g^{(k)}$ multiplicatively

$$g^{(k)} = \zeta(l)^{k-1} g^{(k)*} \tag{26}$$

very close to the fixed-point $\zeta(0) \simeq 1$, then from equation (26)

$$\frac{\mathrm{d} g^{(k)}}{\mathrm{d} l} = g^{(k)*} \frac{\mathrm{d}[\zeta(l)^{k-1}]}{\mathrm{d} l} \simeq g^{(k)*}(k-1)\zeta(l)^{k-2}\frac{\mathrm{d}\zeta}{\mathrm{d} l}. \tag{27}$$

But on the other hand, from equation (24)

$$\frac{\mathrm{d} g^{(k)}}{\mathrm{d} l} = -\epsilon(k-1)g^{(k)*}\zeta^{k-1} - \zeta^k \beta_0^{(k)}\{g^*\} = \zeta^{k-1}(\zeta-1)(k-1)\epsilon g^{(k)*}. \tag{28}$$

Equating the two equations (27) and (28) we obtain

$$\frac{\mathrm{d} ln\,\zeta(l)}{\mathrm{d} l} = (\zeta - 1)\epsilon \tag{29}$$

from which we identify ζ as a relevant scaling field near the fixed-point ($\zeta = 1$) with scaling exponent $\phi_\zeta = \epsilon$.

3.2. Invariant cumulant generating function: a search by symmetry

Instead of looking for the invariant distribution based on the recursion relations (which seems pretty hopeless) we shall base our search on global symmetry considerations. We shall look first at the moment generating function:

$$f(u) = \log \int_0^\infty \mathrm{d} D\, P(D) \mathrm{e}^{-Du} = \sum_{k=1}^\infty (-1)^k \frac{\langle D^k \rangle_c}{k!} u^k. \tag{30}$$

Comparing with equation (5), we have

$$f(0) = 0 \tag{31}$$

$$f'(0) = 1 \tag{32}$$

$$\frac{\mathrm{d}^k f}{\mathrm{d} u^k}\Big|_{u=0} = g^{(k)} k! 2^k. \tag{33}$$

If we can find a function $f(u)$ such that for $2u = (\nabla\phi)^2$ the integral

$$\bar{\mathcal{L}}[f\{\tfrac{1}{2}(\nabla\phi)^2\}] = \int d^d x \{f[\tfrac{1}{2}(\nabla\phi)^2]\} \tag{34}$$

will be invariant under renormalization, then the corresponding $P(D)$ (to be obtained by inverse Laplace transform of $e^{f(u)}$) will also be invariant.

So the problem has been reduced to finding the function $f(u)$ for which $\bar{\mathcal{L}}(f)$ is invariant. This search must be based on symmetry: one has to look for a function $f(u)$ for which the integral in equation (34) is an invariant under a symmetry operation which itself is preserved under the RG iterations (this insight comes from what is known about the role of the symmetry for RG near the lower critical dimension [11]). Since ϕ is a scalar this cannot strictly be an internal symmetry. It should be a symmetry that mixes the order parameter $\phi(x)$ and the coordinates x. The simplest one is to add ϕ as the $(d + 1)$th component of a new vector:

$$x^\mu = (x, \phi(x)). \tag{35}$$

The simplest invariant is then just the total arc length in this space:

$$S[f\{\tfrac{1}{2}(\nabla\phi)^2\}] = \int d^d x \sqrt{1 + (\nabla\phi)^2} \tag{36}$$

which is invariant under rotations in the $d + 1$ space and is, therefore, a natural candidate for $\bar{\mathcal{L}}[f]$ in equation (34).

This type of action has been introduced and studied by Wallace and Zia [13] to model interfacial fluctuation of Ising systems in $d + 1$ dimension (in this picture $\phi(x)$ is the height of the interface, it carries dimensions of length, and the overall action is made dimensionless by giving dimension of L^ϵ to the temperature).

The inverse Laplace transform of $f(u) \sim e^{[-(1+2u)^{1/2}]}$ is the inverse Gaussian distribution [14]. This distribution has two free parameters which may be related to its average and its variance, in terms of which the universal renormalized distribution is

$$P_u(D) = \left(\frac{1}{2\pi\langle\Delta D^2\rangle}\right)^{1/2}\left(\frac{\bar{D}}{D}\right)^{3/2}\exp\left\{-\frac{\bar{D}}{2\langle\Delta D^2\rangle}\frac{(D-\bar{D})^2}{D}\right\}. \tag{37}$$

The form of this distribution remains invariant under the RG flow. The only parameter which changes is the ratio

$$R = \frac{\langle\Delta D^2\rangle}{\bar{D}^2} = \frac{8g^{(2)}}{\bar{D}^2}. \tag{38}$$

Higher cumulants are expressed in terms of R and $\bar{D}$

$$\langle\Delta D^n\rangle_c = (2n - 3)!!\,R^{n-1}\bar{D}^n. \tag{39}$$

The average $\bar{D}$ is a redundant parameter which can be given any value $\bar{D} > 0$.

On the renormalized trajectory the RG flow for R are determined by the equation

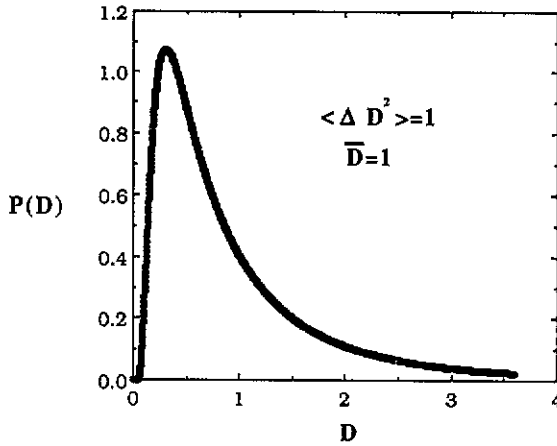$$\frac{dR}{dl} = -\epsilon R + R^2 \tag{40}$$

Figure 1. The form of the invariant inverse Gaussian distribution (equation (37)) with parameters $\check{D} = 1$ and $\langle \Delta D^2 \rangle = 1$ (it is the critical distribution for $d = 1$ within the annealed approximation).

which has $R^* = \epsilon$ as its fixed-point. Thus the fixed-point distribution is (choosing $\check{D} = 1$):

$$P_\epsilon^*(D) = \left(\frac{1}{2\pi D^3 \epsilon}\right)^{1/2} \exp\left\{-\frac{1}{2\epsilon}\frac{(D-1)^2}{D}\right\} \tag{41}$$

this distribution with $\epsilon = 1$ is plotted in figure 1.

For $R < \epsilon$, the RG flow will be to smaller R or $\langle \Delta D^2 \rangle \to 0$ while keeping $\check{D}$ fixed. The flow takes the distribution to

$$\lim_{R\to 0} \left(\frac{\check{D}}{2\pi D^3 R}\right)^{1/2} \exp\left\{-\frac{1}{2R}\frac{(D-\check{D})^2}{D\check{D}}\right\} = \delta(D - \check{D}) \tag{42}$$

as expected.

For $R > \epsilon$, the RG flow will be towards larger R . The moments $\langle \Delta D^k \rangle$ diverge as R^{k-1} and the diffusion process is ill defined. (As $R \to \infty$, $P(D) \sim D^{-3/2}$ for $R^{-1} \ll D \ll R$.)

It is also straightforward to identify $\zeta(l)$ discussed earlier as the scaling field $\zeta(l) = R(l)/R^*$. Hence the crossover exponent for $\Delta R \sim R - R^*$ is ϵ. This is the only relevant direction near the fixed-point [15] and any variation from the fixed-point along other directions is irrelevant, namely the fixed-point is stable in all other directions in this parameter space and the critical manifold (the separatrix) has codimension one. Operators which break the $\phi \to -\phi$ symmetry or which contain power of ϕ, rather than of its gradient, are also relevant [15] but are not important in the present context (besides $m^2\phi^2$ which is discussed next).

3.3. Anomalous diffusion on the critical manifold

For any distribution on the critical manifold the asymptotic scaling behaviour will be determined by the RG flows near the fixed-point. These are expected to be different from the simple diffusion which occurs near the free Gaussian fixed-point.

At the fixed-point the field ϕ^2 will acquire anomalous dimension in order to keep $\bar{D}$ (and all the rest of the couplings) fixed. The anomalous dimension will be exactly that associated with scaling field $\zeta^{-1}(l)$ at the fixed-point may be interpreted as the renormalization factor which multiplies ϕ^2 to keep the couplings fixed. Since $\zeta(l) \sim e^{\epsilon l} = b^{\epsilon}$, the rescaling of ϕ^2 will be

$$\phi^2 \to \zeta^{-1}\phi^2 \sim b^{d-2-\epsilon}\phi^2 \tag{43}$$

which induces an anomalous rescaling of the Laplace transform parameter m^2:

$$m^2 \to m^2 b^{2+\epsilon} \tag{44}$$

which by its definition scales as $1/t$. Hence the time t will rescale as

$$t \to tb^{-2-\epsilon} \tag{45}$$

while length rescales as b^{-1} and, therefore,

$$\langle x^2 \rangle \sim b^{-2} \sim t^{2/(2+\epsilon)}. \tag{46}$$

Hence the diffusion is anomalous with exponent μ in equation (1) given by

$$2\mu = \frac{1}{1 + \frac{1}{2}\epsilon} \sim 1 - \frac{1}{2}\epsilon. \tag{47}$$

Although locally stable we cannot prove that this fixed-point is unique. This question could be explored looking for all possible functions $f((\nabla\phi)^2)$ which are scale invariant. One way to approach the problem is to ask which non-linear symmetries are preserved under RG. The 'Euclidean' symmetry is one of them and the action in equation (36) is its only invariant. We cannot rule out, however, the possibility of other preserved symmetries with other invariants. It will be very interesting if this general question could be investigated more systematically. If other fixed-points exist they may not follow the simple gap scaling above, and a 'multifractal' behaviour cannot be ruled out.

4. The quenched average $(n \to 0)$

Our goal is to pursue the same successful route we followed to the solution in the annealed approximation, for the quenched case as well. We therefore repeat the RG calculation for general n $(n \neq 1)$, and continue it analytically to zero. Here, however, we encounter a new difficulty: the space spanned by the couplings $g^{(k)}$ is not closed under renormalization.

We shall explain it for terms to order $(\partial\phi)^4$ but similar behaviour occurs for higher powers as well (see appendix). The term in the bare Lagrangian is of the form

$$g^{(2,0)}\left(\sum_{\alpha=1}^{n}\sum_{i=1}^{d} \partial_i\phi^{\alpha}\partial_i\phi^{\alpha}\right)^2 = g^{(2,0)} \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{d} \partial_i\phi^{\alpha}\partial_i\phi^{\alpha}\partial_j\phi^{\beta}\partial_j\phi^{\beta}.$$

To order $(g^{(2,0)})^2$ the contraction of two such terms not only give rise to a renormalization of $g^{(2,0)}$ itself but also to the generation of term of the form

$$g^{(0,2)} \sum_{\alpha,\beta} \sum_{i,j} (\partial_i \phi^\alpha \partial_i \phi^\beta)(\partial_j \phi^\alpha \partial_j \phi^\beta). \tag{48}$$

The complete recursion relations are given in the appendix. Clearly such a term takes us outside of the realm of the diffusion model with which we began. If we insist on still following the RG flow for a larger number of unphysical parameter spaces we can generalize the approach based on symmetry discussed earlier for the case $n = 1$ to general n, by looking at a vector in the $d + n$ dimension space $(x_1, \cdots, x_d, \phi^1(x), \cdots, \phi^n(x))$. The invariant area in this space is

$$\mathcal{S}[\phi^\alpha(x)] = \int \mathrm{d}^d x (\det g)^{1/2} \tag{49}$$

with $g_{ij} = \delta_{ij} + \sum_{\alpha=1}^n \partial_i \phi^\alpha \partial_j \phi^\alpha$.

Although no distribution may correspond to this function (since it cannot be a cumulant generating function) there is a non-trivial fixed-point for finite n. The incomplete one-loop analysis of Lowe and Wallace [16] yields $R^* = \epsilon/n$ and will diverge in the $n \to 0$ limit. If this behaviour survives for a larger number of loops, it would imply that even in the larger space the trivial fixed-point is the only one accessible by the RG approach (which still leaves the possibilty of non-perturbative strong coupling behaviour).

5. Conclusions

In view of the new understanding acquired in the quantum diffusion problem on the importance of renormalizing the full conductance distribution [8–10] (rather than the first two moments alone), we have addressed the question of the distribution of the diffusion constant in random classical diffusion. We have chosen here the simplest model for diffusion in disordered media which has $d = 0$ as its critical dimension. Our analysis was performed on the replicated Lagrangian in which the cumulants of the distribution are related to the coupling constants.

We first looked at the annealed approximation for which we could identify the invariant distribution. We have found a non-trivial fixed-point along the renormalized trajectory separating a free Gaussian diffusion fixed-point for small $\langle \Delta D^2 \rangle / \bar{D}^2$ from a regime where the cumulant ratio diverges $\langle \Delta D^{(k+1)} \rangle / \langle \Delta D^{(k)} \rangle \to \infty$. At the fixed-point itself an anomalous diffusion $\langle x^2 \rangle \sim t^{(1-\epsilon)/2}$ was found.

For the quenched average $n \to 0$ our results are so far negative: the theory is not renormalizable because unphysical terms are generated under renormalization. Such behaviour may occur in other random systems near their lower critical dimension. This may also indicate that an even more general approach (for example including the possibilty of replica symmetry breaking) may be necessary. Other investigations of the distributions of the appropriate physical quantities in other random models may shed more light and will be most noteworthy.

Appendix

In this appendix, we present the recursion relations of two lowest-order couplings in equation (49) for the general n approach. As mentioned in section 4, we need to consider other (cross) terms like the one in equation (48) which do not appear in the original Lagrangian in equation (21). Including the generated terms, one should rewrite the Lagrangian as follows

$$
\mathcal{L}[\phi^\alpha(x)] = \int d^d x \frac{1}{2} \sum_{\alpha,i} \partial_i \phi^\alpha \partial_i \phi^\alpha - \sum_{\alpha,\beta} \sum_{i,j} [g^{(2,0)} (\partial_i \phi^\alpha \partial_i \phi^\alpha \partial_j \phi^\beta \partial_j \phi^\beta)
$$

$$
+ g^{(0,2)} (\partial_i \phi^\alpha \partial_i \phi^\beta \partial_j \phi^\alpha \partial_j \phi^\beta)]
$$

$$
- \sum_{\alpha,\beta,\gamma} \sum_{i,j,k} [g^{(3,0)} (\partial_i \phi^\alpha \partial_i \phi^\alpha)(\partial_j \phi^\beta \partial_j \phi^\beta)(\partial_k \phi^\gamma \partial_k \phi^\gamma)
$$

$$
+ g^{(1,2)} (\partial_i \phi^\alpha \partial_i \phi^\alpha)(\partial_j \phi^\beta \partial_j \phi^\gamma)(\partial_k \phi^\beta \partial_k \phi^\gamma)
$$

$$
+ g^{(0,3)} (\partial_i \phi^\alpha \partial_i \phi^\beta)(\partial_j \phi^\gamma \partial_j \phi^\alpha)(\partial_k \phi^\beta \partial_k \phi^\gamma)]
$$

$$
+ \text{higher power terms.} \tag{50}
$$

By diagrammatic calculations, we found the following recursion relations for $g^{(2,0)}$ and $g^{(0,2)}$ (to order one loop):

$$
\frac{dg^{(2,0)}}{dl} = -\epsilon g^{(2,0)} + 12 g^{(3,0)} + (2n+2) g^{(1,2)} + 3 g^{(0,3)} + 40 (g^{(2,0)})^2
$$

$$
+ (16n + 32) g^{(0,2)} g^{(2,0)} + (2n + 14)(g^{(0,2)})^2 \tag{51}
$$

$$
\frac{dg^{(0,2)}}{dl} = -\epsilon g^{(0,2)} + 8 g^{(1,2)} + (3n+6) g^{(0,3)} + (12n+28)(g^{(0,2)})^2
$$

$$
+ 64 g^{(0,2)} g^{(2,0)} + 16 (g^{(2,0)})^2. \tag{52}
$$

For the case $n = 1$, the distinction between the two terms in equation (50) with coefficients $g^{(2,0)}, g^{(0,2)}$ disappears, so one can identify $g^{(2)}$ in equation (22) as the sum of $g^{(2,0)}$ and $g^{(0,2)}$ in equation (50). For the same reason, $g^{(3)}$ in equation (22) is the sum of $g^{(3,0)}$, $g^{(1,2)}$ and $g^{(0,3)}$. Therefore the sum of these two equations is identical to equation (22), for $n = 1$.

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